CYCLIC FIELDS OF DEGREE EIGHT*

BY

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1. **Introduction.** Let F be any non-modular field, C be an algebraic extension of degree n of F. Then $C=F(x)$ is the field of all rational functions with coefficients in F of a root x of an equation $\phi(\omega)=0$ which has coefficients in F, degree n, and transitive group G for F.

The problem of the construction of all equations of degree n and group G is evidently equivalent to the problem of the construction of all corresponding fields C. Moreover the construction of a set of canonical equations $\psi(\omega)=0$ with the property that every $C=F(x)$ of degree n and group G is equal to an $F(y)$ defined by a $\psi(\omega)=0$ provides a solution of both problems.

One of the most important problems in the algebraic theory of fields is the construction of all cyclic fields of degree n over F. This is the case where G consists of the n distinct powers S^i $(i=0, 1, \cdots, n-1)$ of a single substitution S. In this case G is also the group of all automorphisms of C. Moreover this problem has been reduced to the case $n=p^e$, p a prime.

Cyclic fields of degree 2, 2^2 have been constructed.† In the present paper we shall use purely algebraic methods to construct all cyclic fields of degree $2^3=8$‡ over any non-modular field F.

2. **General theory of cyclic fields.** Let F be any non-modular field and let C be a cyclic field of degree n over F. Then if

$$n = p_1^{e_1}\cdot p_2^{e_2}\cdots p_t^{e_t}, \tag{1}$$

where the p_i are distinct primes, it is well known that C is the direct product

$$C = C^{(1)}\times C^{(2)}\times\cdots\times C^{(t)} \tag{2}$$

of cyclic fields $C^{(i)}$ of degree $p_i^{e_i}$ over F. Conversely every direct product (2) is a cyclic field of degree n over F. It is thus certain that the problem of constructing all cyclic fields of degree n over F is equivalent to the corresponding problem for the case $n=p^e$.

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† Cf. §2.

‡ Cyclic fields of degree eight have been considered by F. Mertens in the Wiener Sitzungsberichte, vol. 125 (1916), pp. 741–831. But he considered algebraic number fields, used the arithmetic theory of ideals, and did not give very explicit results. His method is not at all applicable to the case we are considering (where F is a general field). Moreover I believe the results obtained here are more explicit and give a more definite construction for C even for the cases considered by Mertens.

Let then $C=C_e$ have degree $n=p^e$, p a prime. It is well known that we may define a chain of fields

(3) $$C_e > C_{e-1} > \cdots > C_1 > C_0 = F,$$

where C_i is cyclic of degree p^i over F, cyclic of degree p over C_{i-1}. In fact let S be the automorphism of C generating its group G of automorphisms. Then this group of order n is given by

(4) $$G_e = (I, S, S^2, \cdots, S^{n-1}), S^n = I.$$

But if $T=S^{p^{e-1}}$ then

(5) $$H = (I, T, T^2, \cdots, T^{p-1})$$

is an invariant sub-group of G of index p^{e-1} defining a sub-field C_{e-1} of degree p^{e-1} and with Galois group

(6) $$G_{e-1} = (I, \sigma, \sigma^2, \cdots, \sigma^{m-1}), m = p^{e-1},$$

isomorphic with G_e but with $T=S^m$ in G corresponding to the identity of G_{e-1}.

We may now consider every C_e as a cyclic field of degree p over a cyclic field C_{e-1} of degree p^{e-1} to obtain some of the properties of C_e. But if C_e is cyclic of degree p over C_{e-1} which is cyclic of degree p^{e-1} over F, then it is not necessarily true that C_e is cyclic over F. Thus we shall also require a consideration of further properties.

We are interested here only in the case $p=2$. Let C be a cyclic field of degree $n=2^e$, $e>1$ over F, and let D be its uniquely defined sub-field of degree $m=2^{e-1}$. Then C is a quadratic field over D,

(7) $$C = D(x),\ x^2 = a \text{ in } D,$$

where 1, x are linearly independent with respect to D. The substitution S generating the cyclic group of C has order n and D consists of all quantities d of C such that

(8) $$d^{S^m} = d \qquad (m = 2^{e-1}).$$

For convenience of notation we shall write

(9) $$c^{S^k} = c^{(k)},$$

so that $c^{(m)}=c$ whenever c is in D but not otherwise. Then

$$[x^{(m)}]^2 = a^{(m)} = a = x^2,$$

and $x^{(m)} = \pm x$. But x is not in D. Hence

(10) $$x^{(m)} = -x.$$

In particular let $x' = \alpha + \beta x$ where α and β are in D. Then $(x')^2 = a' = \alpha^2 + \beta^2 a + 2\alpha\beta x$. But a' is in D so that $2\alpha\beta = 0$. If $\beta = 0$ then $x' = \alpha$ is in D and $(x')^{(n-1)} = x = \alpha^{(n-1)}$ is in D, a contradiction. Hence $\beta \neq 0$, $\alpha = 0$ and

$$x' = \beta x, \qquad \beta \text{ in } D. \tag{11}$$

It is obvious that $D(x) = D(bx)$ for every non-zero b of D. Hence all of the above properties as well as those we may derive later will hold for any bx taken as the quantity generating C, a quadratic field over D.

We shall assume first that $n = 2$. Then $D = F$ and, since x is not in D, the field $F(x) = D(x) = C$ is a quadratic field over F generated by x. Let next

$$m = 2^{e-1} = 2g, \; g \geqq 1,$$

so that $n \geqq 4$. Then $D = K(y)$, $y^2 = \alpha$ in K, is a quadratic field over the field K of all quantities k of C such that

$$k^{(g)} = k.$$

The field $F(x)$ is a sub-field of $C = D(x)$. But x is not in $D \geqq F(x^2) = F(a)$ so that the degree of $F(x)$ is $2h$ where h is the degree of $F(x^2)$. Hence $F(x) = C$ if and only if $F(a) = D$.

Suppose that $F(a) < D$. Then a is in a proper sub-field of D. But D is cyclic and its maximal proper sub-field K contains every proper sub-field of D. Hence a is in K, $a^{(g)} = a$, $[x^{(g)}]^2 = a^{(g)} = x^2$. Then $x^g = \pm x$, $x^{(m)} = [x^{(g)}]^{(g)} = x$, a contradiction. Hence $F(a) = D$ and we have

THEOREM 1. *Let C be a cyclic field of degree $n = 2m$ over F, $C = D(x)$ where D is a cyclic sub-field of C of degree $m = 2^{e-1}$ over F so that x may be chosen so that*

$$x^2 = a \text{ in } D.$$

Then $x' = \beta \cdot x$ where β is in D and has the property that $x^{(m)} = -x$. Moreover this latter property implies that $F(x) = C$, $F(a) = D$.

Suppose that $x_0 = bx$, $b \neq 0$ in D. Then $x_0{}^2 = b^2 x^2 = b^2 a$ is in D, $x_0{}^{(m)} = b^{(m)} x^m = -bx = -x_0$. By Theorem 1, $C = F(x) = D(x) = D(x_0) = F(bx)$.

THEOREM 2. *Let $x_0 = bx$ where $b \neq 0$ is in D. Then $F(x) = F(x_0) = C$.*

The condition $x' = \beta \cdot x$ imposes two restrictions on β. The first is obviously $x'^2 = \beta^2 \cdot x^2 = (x^2)' = a' = \beta^2 \cdot a$, a necessary and sufficient condition that x' shall actually equal $\beta \cdot x$. Next we must have $x^{(m)} = -x$. But

$$x'' = (\beta x)' = \beta' \beta x, \cdots, x^{(k)} = [x^{(k-1)}]' = [\beta^{(k-1)} \beta^{(k-2)} \cdots \beta' \beta] x,$$

and

$$x^{(m)} = [\beta^{(m-1)} \beta^{(m-2)} \cdots \beta' \beta] x = -x,$$

so if we write

$$N_D(\beta) = \beta\beta'\beta'' \cdots \beta^{(m-1)},$$

then it follows from $x^{(m)} = -x$ that

$$N_D(\beta) = -1. \tag{12}$$

Conversely let D be cyclic of degree $m = 2^{e-1}$ over F and let a, β satisfy (12). Then the field $D(x)$ defined by a root x of $x^2 = a$ is a quadratic field over D if and only if a is not the square of any quantity of D. But if $a = c^2$, c in D, then $\beta^2 = (a')(a)^{-1} = (c'c^{-1})^2$ so that

$$\beta = \pm (c')(c)^{-1}. \tag{13}$$

But m is even and

$$N_D(\beta) = (\pm 1)^m N_D\left(\frac{c'}{c}\right) = (\pm 1)^m = 1, \tag{14}$$

a contradiction of the first equation of (12). Hence $D(x)$ has degree $n = 2m$. Also if we define $x' = \beta \cdot x$ then (12) implies that $x^{(m)} = -x$ so that we have defined a self correspondence of $C = D(x)$

$$c + dx \longleftrightarrow c' + d'x' = c' + d'\beta x, \tag{15}$$

for every c and d of D, $c + dx$ of C. This correspondence is evidently preserved under addition, subtraction, multiplication and division and is an automorphism of C if and only if $x'^2 = a'$ which is satisfied by (12). Hence (15) is an automorphism S of C and, since S^m is an automorphism of C in which x corresponds to $-x$ the order of S is $n = 2m$ and C is a cyclic field. Obviously D is the set of all quantities of C unaltered by S^m. By Theorem 1, $C = F(x)$ and we have proved

THEOREM 3. *Let D be cyclic of degree 2^{e-1} over F with generating automorphism*

$$d \longleftrightarrow d',$$

for every d of D. Then D is the unique sub-field of degree m of a cyclic field C of degree $n = 2m$ if and only if there exist quantities $\beta \neq 0$, $a \neq 0$ in D, such that

$$\beta^2 = \frac{a'}{a}, \quad N_D(\beta) = -1. \tag{16}$$

Moreover every solution of (16) *defines a cyclic field of degree n over F*

$$C = F(x),\ x^2 = a \text{ in } D,\ x \longleftrightarrow x' = \beta \cdot x, \tag{17}$$

as generating automorphism, so that D is the set of all quantities d of C such that $d^{(m)} = d$.

The case $m=1$, $n=2$ is trivial so that we shall assume henceforth that $n>2$, $n=4g=2m$. Then (16) implies that

$$(18)\ \beta^2 = \frac{a'}{a},\quad \beta'^2 = \frac{a''}{a'},\ \cdots,\ (\beta^{(i-1)})^2 = \frac{a^{(i)}}{a^{(i-1)}},\ \cdots,\ (\beta^{(g-1)})^2 = \frac{a^{(g)}}{a^{(g-1)}},$$

and hence that

$$(19)\qquad [\beta\beta'\cdots\beta^{(g-1)}]^2 = \frac{a^{(g)}}{a}\cdot$$

Then if

$$(20)\qquad y = a\beta\beta'\cdots\beta^{(g-1)},$$

equation (19) implies

$$(21)\qquad y^2 = aa^{(g)}.$$

But D is a quadratic field $K(d)$, d^2 in K, over a cyclic field K of degree g over F. Moreover

$$(22)\qquad k^{(g)} = k$$

for every k of K. Since $m=2g$, $a^{(m)}=a$, we have $[aa^{(g)}]^{(g)}=a^{(g)}a$ so that y^2 is in K. Also $yy^{(g)}=aa^{(g)}[\beta\beta'\cdots\beta^{(g-1)}][\beta^{(g)}\cdots\beta^{(m-1)}]=aa^{(g)}N_D(\beta)=-aa^{(g)}$ $=-y^2$, $y^{(g)}=-y$, and y is not in K. But then y generates D, a quadratic field over K, and

$$(23)\qquad D = K(y),\ y^2 = \alpha \text{ in } K.$$

The field $D=K(y)$ is a quadratic field over K which is cyclic of degree g over F. By Theorem 3 there exist quantities $\alpha=y^2$ in K, $\gamma=y'y^{-1}$ in K, such that

$$(24)\qquad \gamma^2 = \frac{\alpha'}{\alpha},\quad N_K(\gamma) \equiv \gamma\gamma'\cdots\gamma^{(g-1)} = -1,$$

and, by this same theorem, $D=F(y)$. Hence

$$(25)\qquad C = F(x),\ x^2 = a \text{ in } D,\ D = F(y),\ y^2 = \alpha \text{ in } K,$$

$$(26)\qquad x' = \beta x,\ y' = \gamma y,\ a = \frac{y}{\beta\beta'\cdots\beta^{(g-1)}}\cdot$$

We now wish

$$(27)\qquad \beta^2 = \frac{a'}{a} = \frac{y'}{\beta'\beta''\cdots\beta^{(g)}}\ \frac{\beta\beta'\cdots\beta^{(g-1)}}{y} = \frac{y'}{y}\ \frac{\beta}{\beta^{(g)}}\cdot$$

But (27) is equivalent to

$$\frac{y'}{y} = \beta\beta^{(g)}, \tag{28}$$

that is, $\gamma = \beta\beta^{(g)}$.

Conversely, let $y^2 = \alpha$ in K, $\gamma^2 = \alpha'\alpha^{-1}$, $N_K(\gamma) = -1$, so that, by Theorem 3, $K(y) = F(y)$ is a cyclic field of degree $2g$ over F. Let also a be defined by the third equation of (26), and β be in D and satisfy

$$\gamma = \beta\beta^{(g)}. \tag{29}$$

Then

$$N_D(\beta) = [\beta\beta' \cdots \beta^{(g-1)}][\beta\beta' \cdots \beta^{(g-1)}]^{(g)} = [\beta\beta^{(g)}][\beta\beta^{(g)}]' \cdots [\beta\beta^{(g)}]^{(g-1)}$$
$$= \gamma\gamma' \cdots \gamma^{(g-1)} = N_K(\gamma) = -1.$$

Also

$$\frac{a'}{a} = \frac{y'}{\beta'\beta'' \cdots \beta^{(g)}} \frac{\beta\beta' \cdots \beta^{(g-1)}}{y} = \frac{\gamma\beta}{\beta^{(g)}} = \frac{\gamma}{\beta\beta^{(g)}} \cdot \beta^2 = \beta^2,$$

as desired. We are now in a position to prove

THEOREM 4. *Let $n = 4g = 2^e$ and let K be a cyclic field of degree g over F with automorphism $k \longleftrightarrow k'$ for every k of K. Then K is the unique sub-field of degree g of a cyclic field C of degree n over F if and only if there exist quantities $\alpha \neq 0$, $\gamma \neq 0$, β_1, β_2 in K satisfying*

$$\gamma^2 = \frac{\alpha'}{\alpha}, \quad N_K(\gamma) = -1, \gamma = \beta_1^2 - \beta_2^2\alpha. \tag{30}$$

Every solution of (30) *defines a cyclic field $F(x) = C$ with*

$$y^2 = \alpha, \beta = \beta_1 + \beta_2 y, x^2 = a = \frac{y}{\beta\beta' \cdots \beta^{(g-1)}}, \tag{31}$$

and with generating automorphism S given by

$$c = c_1 + c_2 y + (c_3 + c_4 y)x \longleftrightarrow c' = c_1' + c_2'\gamma y + (c_3' + c_4'\gamma y)\beta x, \tag{32}$$

for every c_1, c_2, c_3, c_4 of K, c of C, so that

$$x' = \beta x, \; y' = \gamma y. \tag{33}$$

We obviously also will have

COROLLARY 1. *In Theorem 4 the field K is the field of all quantities of C unaltered by S^q, the field $D=F(y)=K(y)$ is the field of all quantities of C unaltered by S^m.*

For we need only notice in the above that, since β is in D, $\beta=\beta_1+\beta_2 y$ where β_1 and β_2 are in K. We have then merely replaced the condition $\gamma=\beta\beta^{(q)}$ by the equivalent condition $\gamma=\beta_1^2-\beta_2^2\alpha$ of (30).

We shall now obtain some important restrictions which it is possible to impose on β. Suppose first that $n=4$, $m=2$. Then $K=F$, $N_K(\gamma)=\gamma=-1$ is in F, and (30) becomes merely $\beta_1^2-\beta_2^2\alpha=-1$. If $\beta_1=0$ then $\beta_2\neq 0$, $\alpha=(\beta_2^{-1})^2$ which is impossible if D is a quadratic field over F. Hence for this case $\beta_1\neq 0$.

There exists the possibility in the above theorem that $\beta_1\beta_2=0$. We shall be able to restrict β so that all fields C are obtained yet $\beta_1\beta_2\neq 0$.

By Theorem 2 if $b=b_1+b_2 y$, $b_1 b_2\neq 0$, b_1 and b_2 in K, then $x_0=bx$ also generates $F(x)$ and satisfies

$$x_0^2=a_0=\frac{y_0}{\beta_0\beta_0'\cdots\beta_0^{(q-1)}},\quad x_0'=\beta_0 x_0,\quad y_0^2=\alpha_0,\quad y_0'=\gamma_0 y_0, \tag{34}$$

with

$$\gamma_0^2=\frac{\alpha_0'}{\alpha_0},\quad N_K(\gamma_0)=-1,\quad \beta_0=\beta_{10}+\beta_{20}y_0,\quad \beta_{10}^2-\beta_{20}^2\alpha_0=\gamma_0. \tag{35}$$

But

$$x_0'=(bx)'=b'\beta x=\beta_0 x_0=\beta_0 bx, \tag{36}$$

$$\begin{aligned}\beta_0&=\frac{b'}{b}\beta=\frac{b_1'+b_2'\gamma y}{b_1+b_2 y}\beta=(b_1'+b_2'\gamma y)(b_1-b_2 y)e\\&=[(b_1'b_1-b_2'b_2\alpha\gamma)+(b_2'\gamma b_1-b_1'b_2)y]e,\end{aligned} \tag{37}$$

where

$$e=(b_1^2-b_2^2\alpha)^{-1}\beta \tag{38}$$

is either in K or a multiple of y by a quantity of K according as $\beta_2=0$ or $\beta_1=0$. But then $\beta_{10}\beta_{20}=0$ if and only if

$$b_2'\gamma b_1-b_1'b_2=0,\ \text{or}\ b_1'b_1-b_2'b_2\alpha\gamma=0. \tag{39}$$

Suppose first that $b_2'\gamma b_1-b_1'b_2=0$. Then since $b_1b_2\neq 0$,

$$\gamma=\frac{b_1'b_2}{b_1b_2'},\quad \gamma'=\frac{b_1''b_2'}{b_1'b_2''},\ \cdots,\ \gamma^{(q-1)}=\frac{b_1^{(q)}b_2^{(q-1)}}{b_1^{(q-1)}b_2^{(q)}} \tag{40}$$

and

$$(41)\qquad -1 = N_K(\gamma) = \frac{b_1'b_2}{b_1b_2'}\cdot\frac{b_1''b_2'}{b_1'b_2''}\cdots\frac{b_1^{(\sigma)}b_2^{(\sigma-1)}}{b_1^{(\sigma-1)}b_2^{(\sigma)}} = \frac{b_1^{(\sigma)}}{b_1}\cdot\frac{b_2}{b_2^{(\sigma)}} = 1,$$

since $b_1=b_1^{(\sigma)}$ and $b_2=b_2^{(\sigma)}$ are in K, a contradiction. Hence $b_2'\gamma b_1-b_1'b_2\neq 0$.

We have then proved that if $\beta_{20}\beta_{10}=0$ then $b_1b_1'=b_2b_2'\alpha\gamma$. If $n=4$ then $m=2$ and we have already shown that $\beta_1\neq 0$. Hence $\beta_{10}\neq 0$ and hence $\beta_2=\beta_{20}=0$. But the coefficient of y in β_0 when $e\neq 0$ is in $F=K$, that is, $\beta_2=0$, is $\beta_{20}=d(b_2'\gamma b_1-b_1'b_2)\neq 0$ as we have shown. It remains only to consider the case $n>4$, $m>2$, $b_1b_1'=\alpha\gamma b_2b_2'$.

Let $y_0=b_2b_1^{-1}y$. Then $F(y_0)=F(y)$,

$$(42)\qquad y_0y_0' = b_2b_2'(b_1b_1')^{-1}\alpha\gamma = 1,\ y_0' = (y_0)^{-1},\ y_0'' = y_0.$$

But the automorphism S of $D=F(y_0)$ replacing y_0 by y_0' has order m. Hence $m=2$, a contradiction. We have proved

THEOREM 5. *Every cyclic field $F(x)$ of degree $n=2^e$ over F with K as cyclic sub-field is generated by an x of Theorem* 4 *with $\beta_1\beta_2\neq 0$ in* (30).

3. **Cyclic quartic fields.** Let $n=4$ so that $K=F$, $g=1$, γ and α are in F. Then $N_K(\gamma)=\gamma=-1$, $\beta_1^2-\beta_2^2\alpha=-1$ for $\beta_1\neq 0$, $\beta_2\neq 0$ in F. Put $\epsilon=\beta_1^{-1}$ and obtain $-\epsilon^2=1-(\beta_2\beta_1^{-1})^2\alpha$, whence if $u=\beta_2\epsilon y$ then $F(u)=F(y)$ and $u^2=\beta_2^2\epsilon^2y^2=(\beta_2\beta_1^{-1})^2\alpha$,

$$(43)\qquad u^2 = 1+\epsilon^2 = \tau \text{ in } F,\ \beta = \frac{1+u}{\epsilon},$$

since $\epsilon\beta=\epsilon(\beta_1+\beta_2y)=1+u$. Also

$$(44)\qquad x^2 = a = \frac{y}{\beta} = \frac{\beta_2\epsilon y}{\beta_2(1+u)} = \frac{u}{\beta_2(1+u)}\cdot\frac{1-u}{1-u} = \nu(u-\tau),$$

where $\nu=(-\beta_2\epsilon^2)^{-1}\neq 0$ is in F. We have therefore proved the well known result

THEOREM 6. *Every cyclic field $F(x)$ of degree four over F is generated by a quantity x satisfying*

$$(45)\qquad x^2 = \nu(u-\tau),\ x' = \frac{1+u}{\epsilon}x,\ u^2 = \tau = 1+\epsilon^2,$$

where ϵ and $\nu\neq 0$ are in F and τ is not the square of any quantity of F.

4. **Cyclic fields of degree eight.** Let now $n=8$, $m=4$, $g=2$. Then $F(y)$ is a cyclic quartic field. We wish $\beta=\beta_1+\beta_2y$ with $y^2=\alpha$ in K, γ, β_1, β_2 in K and $\gamma\gamma'=-1$,

$$(46)\qquad \beta_1^2-\beta_2^2\alpha = \gamma,\ \beta_1\beta_2\neq 0.$$

Let

$$(47)\qquad \delta = \beta y = \beta_2\alpha + \beta_1 y = \delta_1 + y_0,$$

where

$$(48)\qquad F(y_0) = F(y),\ y_0 = \beta_1 y,\ \delta_1 = \beta_2\alpha \text{ in } K.$$

Then

$$(49)\qquad \beta = y^{-1}\delta = (\beta_1 y_0^{-1})\delta,\ (\beta_2\alpha)^2 - \beta_1^2\alpha = -\alpha\gamma,$$

so that, since

$$(50)\qquad y_0^2 = \alpha_0 = \beta_1^2\alpha,\ y_0' = \gamma_0 y_0,$$

we have

$$(51)\qquad \beta = (\beta_1\alpha_0^{-1})\delta y_0 = (\beta_1\alpha_0^{-1})(\alpha_0 + \delta_1 y_0).$$

Also $\beta\beta'\beta''\beta''' = -1$ and hence

$$(52)\qquad a = \frac{y}{\beta\beta'} = -\,y(\beta\beta')'' = -\,y\left(\frac{\delta\delta'}{yy'}\right)'' = -\,\frac{y(\delta\delta')''}{\alpha\gamma}$$

$$= -\,\frac{y_0}{\beta_1}(\beta_1\beta_1')\frac{(\delta\delta')''}{\alpha_0\gamma_0} = \frac{\beta_1'(\delta\delta')''}{-\,\alpha_0\gamma_0}y_0,$$

where

$$(53)\qquad \gamma_0\gamma_0' = -1,\ \gamma_0^2 = \frac{\alpha_0'}{\alpha_0},\ \delta_1^2 - \alpha_0 = -\lambda^{-1}\alpha_0\gamma_0,\ \lambda = \beta_1\beta_1'.$$

Suppose that γ_0 is in F. Then $\gamma_0\gamma_0' = -1$ gives $\gamma_0^2 = -1$, $\gamma_0 = i = (-1)^{1/2}$. Also $\alpha_0' = -\alpha_0$ and if $K = F(u)$ we may take $\alpha_0 = u$. Then the solution of (46) is equivalent to $\delta_1^2 - \alpha_0 = -\lambda^{-1}\alpha_0\gamma_0$ where λ is in F and hence to the solution of $\delta_1^2 = u(1-\lambda^{-1}i)$. But if $\delta_1 = \xi_1 + \xi_2 u$ this implies that $u^2 = \tau$ in F, $\xi_1^2 + \xi_2^2\tau + 2\xi_1\xi_2 u = u(1-\lambda^{-1}i)$, $\xi_1^2 + \xi_2^2\tau = 0$ and $\tau = -(\xi_1\xi_2^{-1})^2 = (i\xi_1\xi_2^{-1})^2$, a contradiction of our hypothesis that $F(u)$ is a quadratic field.

Hence γ_0 is not in F and the hypothesis $\beta_2 \neq 0$ of §3 is satisfied for $F(y_0)$. But then

$$(54)\qquad y_0^2 = \alpha_0 = \nu(u-\tau),\ \frac{y_0'}{y_0} = \gamma_0 = \frac{1+u}{\epsilon},\ u^2 = \tau = 1 + \epsilon^2.$$

Also $-\alpha_0\gamma_0 = -\nu\epsilon^{-1}(u-\tau)(u+1) = \nu\epsilon^{-1}u(\tau-1)$; that is, since $\tau - 1 = \epsilon^2$,

$$(55)\qquad -\,\alpha_0\gamma_0 = \nu\epsilon u.$$

We may now complete our computation (52) of a. We use

$$(\delta\delta')''y_0 = [(\delta_1 + y_0)(\delta_1' + y_0')]''y_0 = (\delta_1 - y_0)(\delta_1' - \gamma_0 y_0)y_0$$
$$= (-\alpha_0 + \delta_1 y_0)(\delta_1' - \gamma_0 y_0) = -(\alpha_0\delta_1' + \delta_1\alpha_0\gamma_0) + (\delta_1\delta_1' + \alpha_0\gamma_0)y_0.$$

Hence

$$a = \frac{\beta_1' u}{\nu\epsilon\tau}[\nu\epsilon u\delta_1' - \nu(u - \tau)\delta_1 + (\delta_1\delta_1' - \nu\epsilon u)y_0], \tag{56}$$

where

$$\delta_1 = \xi_1 + \xi_2 u,\ \delta_1' = \xi_1 - \xi_2 u,\ \beta_1 = \xi_3 + \xi_4 u. \tag{57}$$

Also (51) gives $\beta = \beta_1(-\alpha_0\gamma_0)^{-1}(-\alpha_0\gamma_0 - \delta_1\gamma_0 y_0) = \beta_1(\nu\epsilon u)^{-1}(\nu\epsilon u - \delta_1\gamma_0 y_0)$ and hence

$$\beta = \frac{\beta_1}{\nu\epsilon\tau}\left[\nu\epsilon\tau - \frac{\delta_1}{\epsilon}(u + \tau)y_0\right]. \tag{58}$$

We have proved

THEOREM 7. *Every cyclic field $F(x)$ of degree eight over F is generated by a quantity x satisfying*

$$x^2 = a,\ x' = \beta x, \tag{59}$$

with a and β given by (54), (56), (57), (58) *such that $\nu \neq 0$ in F, $\delta_1 \neq 0$, $\beta_1 \neq 0$, and if*

$$\lambda = \xi_3^2 - \xi_4^2\tau, \tag{60}$$

then

$$\delta_1^2 = \alpha_0 - \lambda^{-1}\alpha_0\gamma_0 = \nu(u - \tau + \lambda^{-1}\epsilon u). \tag{61}$$

The quantity $\delta_1^2 = \xi_1^2 + \xi_2^2\tau + 2\xi_1\xi_2 u$, so that (61) is equivalent to

$$-\nu\tau = \xi_1^2 + \xi_2^2\tau,\ 2\xi_1\xi_2 = \nu(1 + \lambda^{-1}\epsilon). \tag{62}$$

But then $-2\xi_1\xi_2\tau = (-\nu\tau)(1 + \lambda^{-1}\epsilon) = (1 + \lambda^{-1}\epsilon)(\xi_1^2 + \xi_2^2\tau)$, so that, since $\nu \neq 0$, equation (61) is equivalent to

$$\nu = (-\tau)^{-1}(\xi_1^2 + \xi_2^2\tau) \neq 0,\ 1 + \lambda^{-1}\epsilon = \frac{-2\xi_1\xi_2\tau}{\xi_1^2 + \xi_2^2\tau}. \tag{63}$$

The first equation of (63) will be taken to determine ν. The second equation becomes

$$-\epsilon = \lambda\left[1 + \frac{2\xi_1\xi_2\tau}{\xi_1^2 + \xi_2^2\tau}\right] = (\xi_3^2 - \xi_4^2\tau)\frac{(\xi_1^2 + \xi_2^2\tau + 2\xi_1\xi_2\tau)}{\xi_1^2 + \xi_2^2\tau}, \tag{64}$$

to be solved for $\xi_1^2+\xi_2^2\tau\neq 0$. But $\xi_1^2+\xi_2^2\tau+2\xi_1\xi_2\tau=(\xi_1+\xi_2\tau)^2+\xi_2^2\tau(1-\tau)$ $=(\xi_1+\xi_2\tau)^2-(\xi_2\epsilon)^2\tau$. Hence if

$$(65) \qquad k = \eta_1 + \eta_2 u = \frac{(\xi_3 + \xi_4 u)(\xi_1 + \xi_2\tau + \xi_2\epsilon u)}{\xi_1^2 + \xi_2^2\tau},$$

where η_1 and η_2 are then explicitly determined in terms of ξ_1, ξ_2, ξ_3, ξ_4, then

$$(66) \qquad kk' = \frac{(\xi_3^2 - \xi_4^2\tau)(\xi_1^2 + \xi_2^2\tau + 2\xi_1\xi_2\tau)}{(\xi_1^2 + \xi_2^2\tau)^2} = -\frac{\epsilon}{\xi_1^2 + \xi_2^2\tau},$$

so that, since $kk'=\eta_1^2-\eta_2^2\tau$,

$$(67) \qquad -\epsilon = (\xi_1^2 + \xi_2^2\tau)(\eta_1^2 - \eta_2^2\tau) \neq 0,$$

where we use $\tau=1+\epsilon^2\neq 1$.

Conversely let $\epsilon\neq 0$ satisfy (67) and define k by $k=\eta_1+\eta_2 u$. Define

$$(68) \qquad \beta_1 = \frac{(\xi_1^2 + \xi_2^2\tau)k}{\xi_1 + \xi_2\tau + \xi_2\epsilon u} = \xi_3 + \xi_4 u,\ r = \xi_1 + \xi_2\tau + \xi_2\epsilon u,$$

where β_1 exists since $\xi_1^2+\xi_2^2\tau\neq 0$ and hence $r\neq 0$. Then we have

$$(69) \qquad -\epsilon = kk'(\xi_1^2 + \xi_2^2\tau) = \frac{\beta_1\beta_1' rr'}{\xi_1^2+\xi_2^2\tau} = \frac{(\xi_3^2 - \xi_4^2\tau)(\xi_1^2 + \xi_2^2\tau + 2\xi_1\xi_2\tau)}{\xi_1^2 + \xi_2^2\tau},$$

and (64) will be satisfied. Moreover if we define ν by $(63)_1$ then (61) will be satisfied. Also $\tau=1+\epsilon^2$ must not be the square of any quantity of F if $F(u)$, $u^2=\tau$, is a quadratic field over F as we are supposing. We have proved

THEOREM 8. *The solution of* (61) *is equivalent to the determination of* ν *by*

$$(70) \qquad \nu = (-\tau)^{-1}(\xi_1^2 + \xi_2^2\tau),$$

and the solution of

$$(71) \qquad -\epsilon = (\eta_1^2 - \eta_2^2\tau)(\xi_1^2 + \xi_2^2\tau)$$

for ϵ, η_1, η_2, ξ_1, ξ_2 *in* F *and such that* $\tau=1+\epsilon^2$ *is not the square of any quantity of* F.

5. **The formulas for α_0, γ_0, a, β.** We have seen how every cyclic field $F(x)$ of degree eight over F is generated by a quantity x such that $x^2=a$, $x'=\beta x$ where a and β are given by (56), (58), (54), (57) as soon as ν, ϵ, $\tau=1+\epsilon^2$, $\beta_1=\xi_3+\xi_4 u$, $\delta_1=\xi_1+\xi_2 u$ have been determined to satisfy (61). We have also shown that the solution of (61) is equivalent to (70) and the solution of the equation (71) with variables in F. Hence we have merely to solve (71), obtaining formulas with parameters for ϵ, η_1, η_2, ξ_1, ξ_2, obtain formulas for

ξ_3 and ξ_4 by the use of (68), and by the substitution of values so obtained in (54), (56), (57), (58) obtain explicit $a, \beta, \alpha_0, \gamma_0$. But the formulas so obtained would be undesirable because of complexity. Hence we shall confine our further work to a consideration of the only remaining non-trivial part of our problem, the solution of (71). Explicit fields of degree eight may then be obtained by carrying out the above work of substitution for every special case.

6. **The case i in F.** Suppose that F contains a quantity i such that $i^2=-1$. Then if $\tau=1+\epsilon^2$, ϵ in F, we wish to solve $-\epsilon=(\xi_1^2+\xi_2^2\tau)(\eta_1^2-\eta_2^2\tau)$ for $\xi_1, \xi_2, \eta_1, \eta_2$ in F and τ not the square of any quantity of F. Let

$$(72) \qquad k_1 = \xi_1 + \xi_2 iu, \; k_2 = \eta_1 + \eta_2 u,$$

so that k_1 is in $F(u)$, $u^2=1+\epsilon^2$, k_2 is in $F(u)$. Then if

$$(73) \qquad k_3 = k_1 k_2 = \lambda + \mu u, \; \lambda, \mu \text{ in } F,$$

we have

$$(74) \qquad \lambda^2 - \mu^2\tau = -\epsilon,$$

since if $k_3'=\lambda-\mu u$ then $k_3k_3' = k_1k_1' \cdot k_2k_2' = [\xi_1^2-(\xi_2 i)^2\tau]\,[\eta_1^2-\eta_2^2\tau] = (\xi_1^2+\xi_2^2\tau)\,(\eta_1^2-\eta_2^2\tau) = -\epsilon$.

Conversely let λ, μ be a solution of (74). Then if k_3 is defined by (73) we have

$$k_1 = (\xi_1 + \xi_2 iu) = \frac{k_3}{k_2} = \frac{\lambda+\mu u}{\eta_1+\eta_2 u} = \frac{(\lambda\eta_1 - \eta_2\mu\tau)}{\eta_1^2 - \eta_2^2\tau} + \frac{\mu\eta_1 - \lambda\eta_2}{\eta_1^2 - \eta_2^2\tau}u,$$

so that

$$(75) \qquad \xi_1 = \frac{\lambda\eta_1 - \mu\eta_2\tau}{\eta_1^2 - \eta_2^2\tau}, \qquad \xi_2 = \frac{(\mu\eta_1 - \lambda\eta_2)}{\eta_1^2 - \eta_2^2\tau}(-i),$$

where η_1 and η_2 not both zero range independently over all quantities of F so that $\eta_1^2-\eta_2^2\tau\neq 0$. We have therefore

THEOREM 9. *Let i be in F, $i^2=-1$, and λ, μ, ϵ range over all solutions of*

$$(76) \qquad \lambda^2 - \mu^2\tau = -\epsilon$$

in F such that $1+\epsilon^2=\tau$ is not the square of any quantity of F. Then every cyclic field of degree eight over F is given by (70), (68), (65), (59), (54), (56), (57), (58) *for every η_1, η_2 not both zero and in F.*

We therefore have only to solve (76). Suppose first that $\mu=0$. Then $\epsilon=-\lambda^2$ and we have proved

THEOREM 10. *Let λ range over all quantities of F such that $1+\lambda^4$ is not the square of any quantity of F. Then* (76) *is satisfied by $\mu=0$, $\epsilon=-\lambda^2$, and defines corresponding cyclic fields.*

Next let $\mu\neq 0$. Define

$$\mu^{-1} = 2\sigma,\ \lambda\mu^{-1} = \rho, \tag{77}$$

so that

$$-\epsilon\mu^{-2} = -4\sigma^2\epsilon = \rho^2 - \tau = \rho^2 - (1+\epsilon^2), \tag{78}$$

$$(\epsilon - 2\sigma^2)^2 - \rho^2 = 4\sigma^4 - 1, \tag{79}$$

and

$$(\epsilon - 2\sigma^2 - \rho)(\epsilon - 2\sigma^2 + \rho) = 4\sigma^4 - 1.$$

Here again we must separate our work into two special cases.

Suppose first that $\epsilon-2\sigma^2-\rho=0$. Then $4\sigma^4=1$, $(2\sigma^2)^2=1$, so that $2\sigma^2=\pm 1$. Moreover if $2\sigma^2=1$ then $(2\sigma)^2=2$, $2\sigma=\mu^{-1}=\pm 2^{1/2}$ so that, since $\lambda=\rho\mu$, we have $\rho=\epsilon-2\sigma^2=\epsilon-1$,

$$\mu = \pm\frac{2^{1/2}}{2},\quad \lambda = (\epsilon-1)\left(\pm\frac{2^{1/2}}{2}\right), \tag{80}$$

and ϵ ranges over all quantities of F such that $1+\epsilon^2$ is not the square of any quantity of F. Moreover if $2\sigma^2=-1$ then $\mu^{-1}=\pm 2^{1/2}i$, $\rho=\epsilon-2\sigma^2=\epsilon+1$, $\lambda=\rho\mu$,

$$\mu = \pm\frac{2^{1/2}i}{2},\quad \lambda = (\epsilon+1)\left(\pm\frac{2^{1/2}i}{2}\right). \tag{81}$$

We have therefore proved

THEOREM 11. *Let ϵ range over all quantities of F such that $1+\epsilon^2$ is not the square of any quantity of F. Then if i is in F, $i^2=-1$, and λ, μ are given by either* (80) *or* (81), *so that $2^{1/2}$ is in F, the condition $\lambda^2-\mu^2\tau=-\epsilon$ is satisfied, and Theorem 9 defines a set of corresponding cyclic fields of degree eight over F.*

Suppose finally that $\epsilon-2\sigma^2-\rho=\pi\neq 0$. Then $\epsilon-2\sigma^2+\rho=(4\sigma^4-1)\pi^{-1}$ and $2(\epsilon-2\sigma^2)=\pi+(4\sigma^4-1)\pi^{-1}$ while $2\rho=(4\sigma^4-1)\pi^{-1}-\pi$. Also $\lambda=\rho\mu$,

$$\epsilon = \frac{(\pi+2\sigma^2)^2-1}{2\pi},\quad \lambda = \frac{4\sigma^4-\pi^2-1}{4\sigma\pi},\ \mu = (2\sigma)^{-1}, \tag{82}$$

and we have proved

THEOREM 12. *Let F contain a quantity i such that $i^2=-1$. Then every cyclic field of degree eight over F is given by Theorem 9 with λ, μ, ϵ determined by either Theorem 10 or 11 or by (82) as $\pi\neq 0$ in F, $\sigma\neq 0$ in F range over all quantities of F such that $\tau=1+\epsilon^2$ is not the square of any quantity of F.*

7. **The case $\tau=-t^2$, t in F.** Let $\tau=-t^2$ where t is in F. Then F contains no quantity i such that $i^2=-1$ since otherwise $\tau=(it)^2$ contrary to the fundamental assumption of our work, namely that $F(u)$, $u^2=\tau$, shall be a quadratic field over F. We wish to solve

$$-\epsilon=[\xi_1^2-(\xi_2 t)^2][\eta_1^2-\eta_2^2\tau], \tag{83}$$

that is, since $\eta_1^2-\eta_2^2\tau\neq 0$,

$$\xi_1^2-(\xi_2 t)^2=\frac{-\epsilon}{\eta_1^2-\eta_2^2\tau}=R\neq 0,\ -1=\epsilon^2+t^2. \tag{84}$$

Since $\epsilon\neq 0$ we evidently have $\xi_1-\xi_2 t=\pi\neq 0$. Then $\xi_1+\xi_2 t=\pi R^{-1}$ so that

$$\xi_1=\frac{\pi^2+R}{2\pi},\quad \xi_2=\frac{R-\pi^2}{2t\pi},\quad R=\frac{-\epsilon}{\eta_1^2-\eta_2^2\tau}, \tag{85}$$

and we have proved

THEOREM 13. *Let ϵ and t range over all quantities of F such that $-1=\epsilon^2+t^2$ and $1+\epsilon^2=\tau$ is not the square of any quantity of F. Then i is not in F, $i^2=-1$, and every cyclic field of degree eight over F is given by* (68), (65), (59), (54), (56), (57), (58), (85) *when η_1 and η_2 not both zero, $\pi\neq 0$ range independently over all quantities of F.*

8. **The case $\tau\neq -t^2$, i not in F.** Let -1 be not the square of any quantity of F and let $K=F(i)$, $i^2=-1$, so that $F(i)$ is a quadratic field over K. Our *only remaining case* is the case $-\tau\neq t^2$ for any t of F. This is sufficient to secure the fact that $K(u)$, $u^2=\tau$, is a quadratic field over K,* that is, $F(i, u)$ is a quartic field over F.

For otherwise let $\tau=z^2$, $z=z_1+z_2 i$ where z_1 and z_2 are in F. Then $\tau=z_1^2-z_2^2+2z_1z_2 i$ so that $z_1z_2=0$. But $\tau\neq z_1^2$ in F, by hypothesis. Hence $z\neq z_1$, $z_2\neq 0$ and $z_1=0$. Then $\tau=(z_2 i)^2=-z_2^2$ contrary to hypothesis. We have therefore proved that τ is not the square of any quantity of K, $K(u)$ is a quadratic field over K. We shall now prove

LEMMA. *Let λ and μ be in $K=F(i)$ so that we may write $\lambda=\lambda_1+\lambda_2 i$, $\mu=\mu_1+\mu_2 i$ with λ_1, λ_2, μ_1, μ_2 in F. Let*

$$\lambda^2-\mu^2\tau=-\epsilon, \tag{86}$$

* This is of course not the field K of preceding sections.

where $\epsilon \neq 0$ *is in* F, $\tau = 1 + \epsilon^2$. *Then*

$$\lambda_1\lambda_2 = \tau\mu_1\mu_2, \tag{87}$$

and there exist quantities η_1, η_2 *in* F *and not both zero such that*

$$\lambda_1\eta_2 = \mu_1\eta_1,\ \lambda_2\eta_1 = \mu_2\tau\eta_2,\ \eta_1^2 - \eta_2^2\tau \neq 0. \tag{88}$$

For $-\epsilon = \lambda^2 - \mu^2\tau = [\lambda_1^2 - \lambda_2^2 + \tau(\mu_2^2 - \mu_1^2)] + 2(\lambda_1\lambda_2 - \mu_1\mu_2\tau)i$. Since $-\epsilon$ is in F and i is not in F we have $\lambda_1\lambda_2 - \mu_1\mu_2\tau = 0$ as desired. If $\lambda_1 \neq 0$, $(88)_1$ is satisfied by $\eta_2 = (\lambda_1^{-1}\mu_1)\eta_1$ for every $\eta_1 \neq 0$ of F and

$$\lambda_2\eta_1 - \eta_2\mu_2\tau = [\lambda_2 - (\lambda_1^{-1}\mu_1\mu_2\tau)]\eta_1 = \lambda_1^{-1}\eta_1[\lambda_1\lambda_2 - \mu_1\mu_2\tau] = 0$$

so that (88) is completely satisfied. If $\lambda_1 = 0$, $\mu_2 \neq 0$, then $\mu_1\mu_2\tau = \lambda_1\lambda_2 = 0$ so that $\mu_1 = 0$ and $(88)_1$ is satisfied. Then (88) is satisfied for every $\eta_1 \neq 0$ in F when we take $\eta_2 = (\mu_2\tau)^{-1}\eta_1\lambda_2$. Hence finally let $\lambda_1 = \mu_2 = 0$. Then (88) is merely $\mu_1\eta_1 = \lambda_2\eta_1 = 0$ which is satisfied for any $\eta_2 \neq 0$ in F and by $\eta_1 = 0$. Also $\epsilon \neq 0$ so that, by (86), $\lambda = \lambda_2 i$ and $\mu = \mu_1$ are not both zero, so that necessarily $\eta_1 = 0$.

Consider now the problem of determining a general solution of (71). Suppose we have a solution and then put

$$k_1 = \xi_1 + (\xi_2 i)u,\ k_2 = \eta_1 + \eta_2 u,\ k_3 = \lambda + \mu u = k_1 k_2. \tag{89}$$

Equation (89) implies $k_3k_3' = \lambda^2 - \mu^2\tau = -\epsilon = k_1k_1'k_2k_2' = (\xi_1^2 + \xi_2^2\tau)(\eta_1^2 - \eta_2^2\tau)$ and (86) is satisfied where

$$\lambda = \xi_1\eta_1 + \xi_2\eta_2\tau i,\ \mu = \xi_1\eta_2 + \xi_2\eta_1 i. \tag{90}$$

But ϵ is in F and, by the above lemma, $\lambda_1\lambda_2 = \tau\mu_1\mu_2$. Also $\lambda_1 = \xi_1\eta_1$, $\lambda_2 = \xi_2\eta_2\tau$, so that $\lambda_1\eta_2 - \mu_1\eta_1 = \xi_1(\eta_1\eta_2 - \eta_2\eta_1) = 0$, $\lambda_2\eta_1 - \lambda_4\tau\eta_2 = \xi_2\tau(\eta_1\eta_2 - \eta_2\eta_1) = 0$, and (88) is satisfied. Hence every solution of (71) defines a solution of (86) in K for which (87) and (88) are satisfied.

Conversely let (86) be satisfied. By the above lemma, (87), (88) are satisfied. Let η_1, η_2 range over all solutions in F of (88), not both zero, and define k_1, k_2, k_3 by (89) so that if

$$k_1 = \frac{k_3}{k_2} = \frac{\lambda_1 + \mu_1 u}{\eta_1 + \eta_2 u} + \frac{\lambda_2 + \mu_2 u}{\eta_1 + \eta_2 u} i,$$

then

$$\xi_1 = \frac{(\lambda_1\eta_1 - \mu_1\eta_2\tau) + (\mu_1\eta_1 - \lambda_1\eta_2)u}{\eta_1^2 - \eta_2^2\tau} = \frac{\lambda_1\eta_1 - \mu_1\eta_2\tau}{\eta_1^2 - \eta_2^2\tau}$$

is in F by (88). Also

$$\xi_2 u = \frac{\lambda_2 + \mu_2 u}{\eta_1 + \eta_2 u} = \frac{(\lambda_2\eta_1 - \mu_2\eta_2\tau) + (\mu_2\eta_1 - \lambda_2\eta_2)u}{\eta_1^2 - \eta_2^2\tau} = \frac{\mu_2\eta_1 - \lambda_2\eta_2}{\eta_1^2 - \eta_2^2\tau} u,$$

and ξ_2 is in F by (88). Hence (86) determines a set of solutions of (71) and we have proved

THEOREM 14. *Let F contain no quantity i such that $i^2=-1$ and let $\epsilon\neq 0$, λ, μ range over all quantities of $K=F(i)$ such that $\lambda^2-\mu^2\tau=-\epsilon$, ϵ is in F, and $\tau=1+\epsilon^2$, $\pm\tau$ is not the square of any quantity of F. Then if we determine all quantities η_1, η_2 satisfying* (88) *and define $C=F(x)$ by* (55)-(59), (65), (68) *we obtain all cyclic fields C of degree eight over F.*

We therefore need only solve (86). This has already been accomplished in §6. Hence we have, without further proof,

THEOREM 15. *Let t range over all quantities of F such that $\pm(1+t^4)$ is not the square of any quantity of F. Then if $\epsilon=-\lambda^2$, $\lambda=t$ or ti, $\mu=0$, we obtain a solution of $\lambda^2-\mu^2\tau=-\epsilon$ and hence a set of cyclic fields of degree eight over F by the use of Theorem* 14.

Next utilize the *proof* of Theorem 11. If $2\mu=\pm 2^{1/2}$, then either μ is in F and $2^{1/2}$ is in F or $2\mu=\pm ti$, $-t=2^{1/2}i$ is in F, $(-2)^{1/2}$ is in F. Similarly if $2\mu=\pm 2^{1/2}i$ then again $2\mu=\pm t$, ti and either $2^{1/2}$ is in F or $(-2)^{1/2}$ is in F.

THEOREM 16. *Let ϵ range over all quantities of F such that $\pm(1+\epsilon^2)=\pm\tau$ is not the square of any quantity of F and let either $2^{1/2}$ or $(-2)^{1/2}$ be in F but $i=(-1)^{1/2}$ be not in F. Then if either* (80) *or* (81) *is satisfied and λ and μ so defined in $K=F(i)$ we obtain a set of cyclic fields of degree eight over F by the use of Theorem* 14.

We finally use Theorem 12 to state immediately

THEOREM 17. *Let F contain no quantity i, $i^2=-1$. Then every cyclic field of degree eight over F is a cyclic field of Theorems* 13, 15, *or* 16 *or is given by Theorems* 9, 14, *with* (82) *satisfied as $\pi\neq 0$, $\sigma\neq 0$ range over all quantities of $F(i)$ such that ϵ is in F and $\pm(1+\epsilon)^2$ is not the square of any quantity of F.*

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